On a multiscale representation of images as hierarchy of edges

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Images as L^2 objects

$f \in L^2(I\!R^2)$, (grayscale) $\mathbf{f} = (f_1, f_2, f_3) \in L^2(I\!R^2)^3$, (colored)

Noticeable features: edges.
well quantified in the smaller BV class

Homogeneous regions, oscillatory patterns of noise or texture.

Images in intermediate spaces 'between' $L^2(I\!R^2)$ and $BV(I\!R^2)$

Q. Images in intermediate spaces 'between' $BV^*(I\!R^2)$ and $BV(I\!R^2)$?

- Intermediate spaces realized by interpolation

Starting with $Y \subset X$, to form a scale of intermediate spaces, $(X, Y)_\theta$, ranging from $(X, Y)_{\theta=0} = X$ to $(X, Y)_{\theta=1} = Y$.

Regularity functionals

- The K functional: $K(f,\eta;X,Y) := \inf_{u+v=f}\Big\{\|v\|_X + \eta\|u\|_Y\Big\}$.

  The behavior of $K(\cdot,\eta)$ as $\eta \downarrow 0$:

  $$(X,Y)_\theta := \{f \;\Big|\; \eta^{-\theta}K(f,\eta;X,Y) \leq Const\}$$

  $$\text{Lorentz} - \text{scale}: (X,Y)_{\theta,q} := \eta^{-\theta}K(f,\eta;X,Y) \in L^q(d\eta/\eta)$$

- Image processing: The J functional

  intermediate spaces between $X = L^2(\Omega)$ and $Y = BV(\Omega)$

  $$J(f,\lambda) := \inf_{u+v=f}\Big\{\lambda\|v\|^2_{L^2} + \|u\|_{BV}\Big\}.$$

  $J(f,\lambda)$ measures how well an L^2 object can be approximated by its BV features, $J(f,\lambda) \sim \lambda^\theta$ as $\lambda \uparrow \infty$

- Rudin-Osher-Fatemi: $[u_\lambda, v_\lambda] = \text{arginf}\, J(f,\lambda)$;
  λ as a *fixed* threshold for cutting out the noisy part of f

Hierarchical (BV, L^2) decomposition

$$f = u_\lambda + v_\lambda, \quad [u_\lambda, v_\lambda] = \underset{u+v=f}{\arg\inf}\, J(f, \lambda; BV, L^2). \tag{1}$$

- u_λ extracts the *edges*; v_λ captures *textures*

- Distinction is scale dependent – 'texture' at a λ-scale consists of significant edges when viewed under a refined 2λ-scale

$$v_\lambda = u_{2\lambda} + v_{2\lambda}, \quad [u_{2\lambda}, v_{2\lambda}] = \underset{u+v=v_\lambda}{\arg\inf}\, J(v_\lambda, 2\lambda). \tag{2}$$

- A better two-scale representation: $f \approx u_\lambda + u_{2\lambda}$

This process can continue...

Hierarchical (BV, L^2) decomposition...

Starting with an initial scale $\lambda = \lambda_0$,

$$f = u_0 + v_0, \quad [u_0, v_0] = \arg\inf_{u+v=f} J(f, \lambda_0)$$

successive application of dyadic refinement: $\lambda_j := \lambda_0 2^j$

$$v_j = u_{j+1} + v_{j+1}, \quad [u_{j+1}, v_{j+1}] := \arg\inf_{u+v=v_j} J(v_j, \lambda_{j+1}), \; j = 0, 1, ...,$$

After k hierarchical step:

$$\begin{aligned} f &= u_0 + v_0 = \\ &= u_0 + u_1 + v_1 = \\ &= \ldots\ldots \qquad = \\ &= u_0 + u_1 + \cdots + u_k + v_k. \end{aligned}$$

Hierarchical (BV, L^2) decomposition...

- A description of f in an intermediate scale of spaces (BV, L^2)

$$f \sim \sum_{j=1}^{k} u_j + v_k$$

- The multi-layered (BV, L^2) expansion, $f \sim \sum_j u_j$

  particularly suitable for image representations

- Applications of multi-layered representations:

  [Meyer,Averbuch,Coifman]

- Multi-layered representation is

  (i) *hierarchical* and (ii) *essentially nonlinear*

Convergence of the (BV, L^2) expansion

Compare the minimizer $v_j = u_{j+1} + v_{j+1}$ vs. the trivial $[0, v_j]$:

$$J(v_j, \lambda_{j+1}) = \|u_{j+1}\|_{BV} + \lambda_{j+1}\|v_{j+1}\|_2^2 \leq \lambda_{j+1}\|v_j\|_2^2.$$

u_j's capture the BV scale $\sim \lambda_j := \lambda_0 2^j$: $\quad \sum_j \frac{1}{\lambda_j}\|u_j\|_{BV} \leq \|f\|_2^2$

$$f = \sum_{j=0}^{\infty} u_j : \qquad \|f - \sum_{j=0}^{k} u_j\|_{W^{-1,\infty}} = \|v_{k+1}\|_{W^{-1,\infty}} = \frac{1}{\lambda_{k+1}},$$

- The geometric convergence rate is *universal*
- Energy decomposition: $\displaystyle\sum_{j=0}^{\infty}\left[\frac{1}{\lambda_j}\|u_j\|_{BV} + \|u_j\|_2^2\right] = \|f\|_2^2$

key observation: squaring the refinement step, $v_{j+1} + u_{j+1} = v_j$,

$$2(u_{j+1}, v_{j+1}) + \|u_{j+1}\|_2^2 + \|v_{j+1}\|_2^2 = \|v_j\|_2^2,$$

$$\Longrightarrow \quad \frac{1}{\lambda_j}\|u_{j+1}\|_{BV} + \|u_{j+1}\|_2^2 = \|v_j\|_2^2 - \|v_{j+1}\|_2^2, \qquad j = -1, 0, 1, \ldots$$

Initialization

- λ_0 should capture smallest scale in f: $\dfrac{1}{2\lambda_0} \le \|f\|_{W^{-1,\infty}} \le \dfrac{1}{\lambda_0}$

- To capture the missing larger scales:

$$v_j = u_{j-1} + v_{j-1}, \quad [u_{j-1}, v_{j-1}] := \underset{u+v=v_j}{\arg\inf}\ J(v_j, \lambda_{j-1}),\ j = 0, -1, ...,$$

- Running through smaller scales, $\lambda_j = \lambda_0 2^j,\ j \downarrow$:

  Exhaust the oscillatory part of f: $\lambda_0 2^{-k_0}\|f\|_{W^{-1,\infty}} \le 1$.

$$\begin{aligned} v_0 &= u_{-1} + v_{-1} = \\ &= u_{-1} + u_{-2} + v_{-2} = \\ &= \ldots\ldots\ \ldots \qquad = \\ &= u_{-1} + u_{-2} + \ldots\ldots + u_{-k_0}. \end{aligned}$$

The multiscale (BV, L^2) expansion now reads: $\boxed{f = \sum_{j=-k_0}^{\infty} u_j}$

Numerical discretization

- Euler-Lagrange equations:

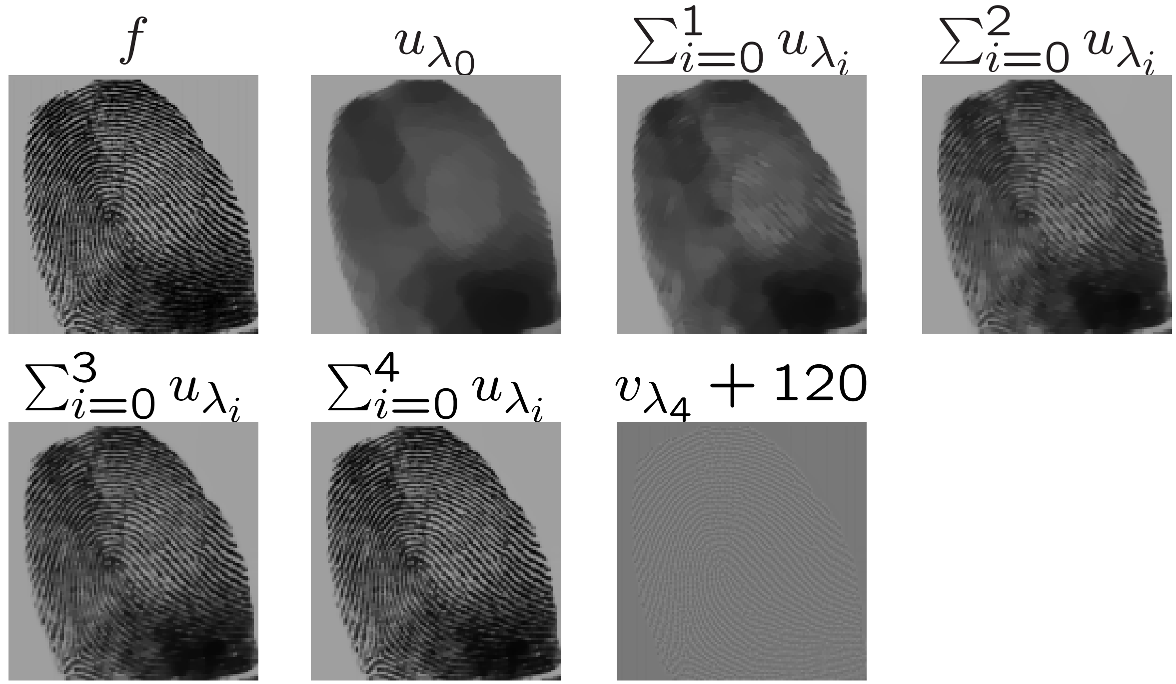
$$u_\lambda - \frac{1}{2\lambda}\text{div}\left(\frac{\nabla u_\lambda}{|\nabla u_\lambda|}\right) = f, \qquad \frac{\partial u_\lambda}{\partial n}_{|\partial\Omega} = 0$$

Fixed point Gauss-Seidel iterations to solve
$$\boxed{\begin{array}{l} f_{new} \longleftarrow f_{current} - u_\lambda \\ \lambda_{new} \longleftarrow 2\lambda_{current} \end{array}}$$

$$\begin{aligned} u_{i,j} &= f_{i,j} + \frac{1}{2\lambda} D_{-x}\left[\frac{1}{\sqrt{\varepsilon^2 + (D_{+x}u_{i,j})^2 + (D_{0y}u_{i,j})^2}} D_{+x}u_{i,j}\right] \\ &\quad + \frac{1}{2\lambda} D_{-y}\left[\frac{1}{\sqrt{\varepsilon^2 + (D_{0x}u_{i,j})^2 + (D_{+y}u_{i,j})^2}} D_{+y}u_{i,j}\right] \end{aligned}$$

$$\boxed{u_{j+1} - \frac{1}{2\lambda_{j+1}}\text{div}\left(\frac{\nabla u_{j+1}}{|\nabla u_{j+1}|}\right) = -\frac{1}{2\lambda_j}\text{div}\left(\frac{\nabla u_j}{|\nabla u_j|}\right)}$$

Fingerprint

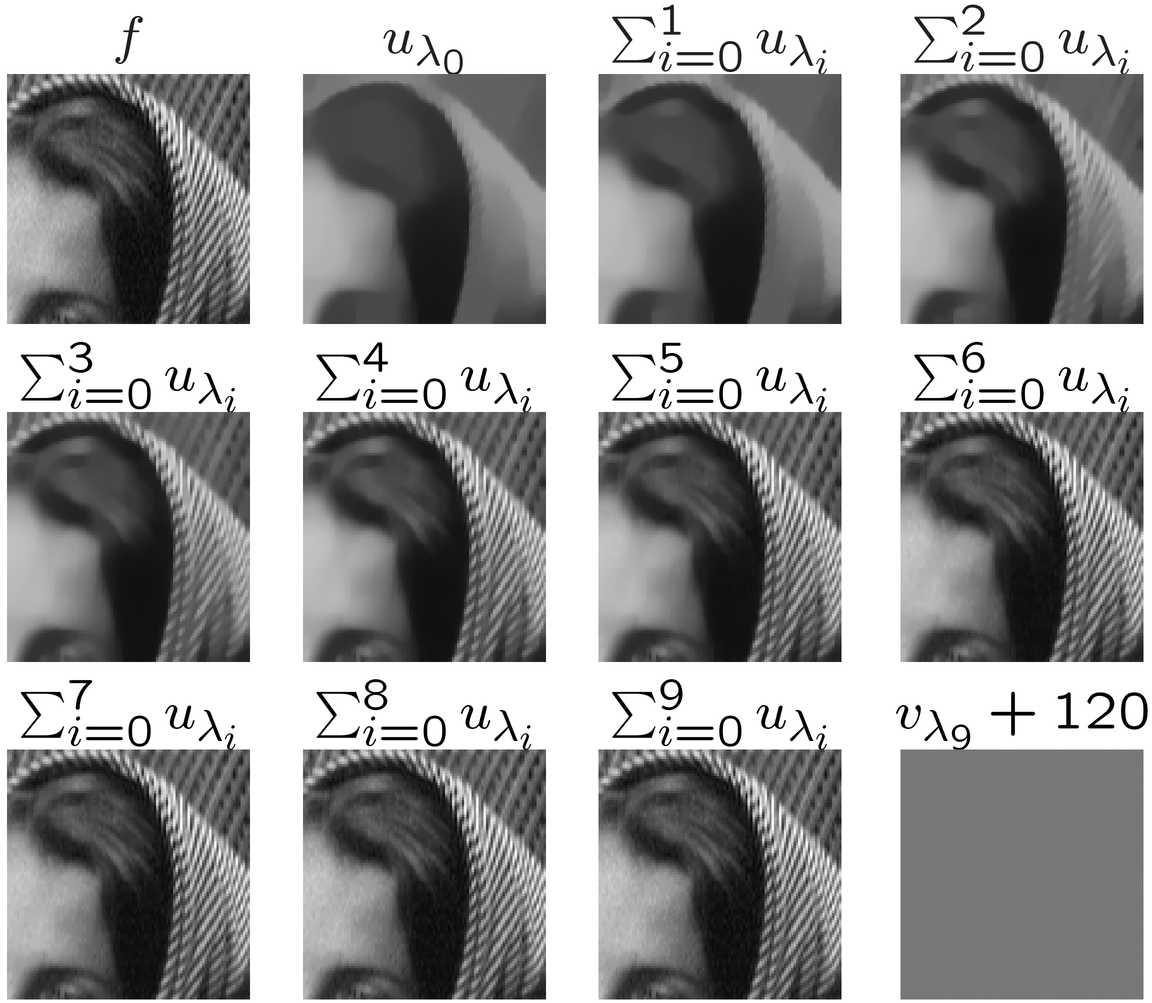


Decomposition of an initial image of a fingerprint for 5 steps with $\lambda_0 = .01$.

Barbara I

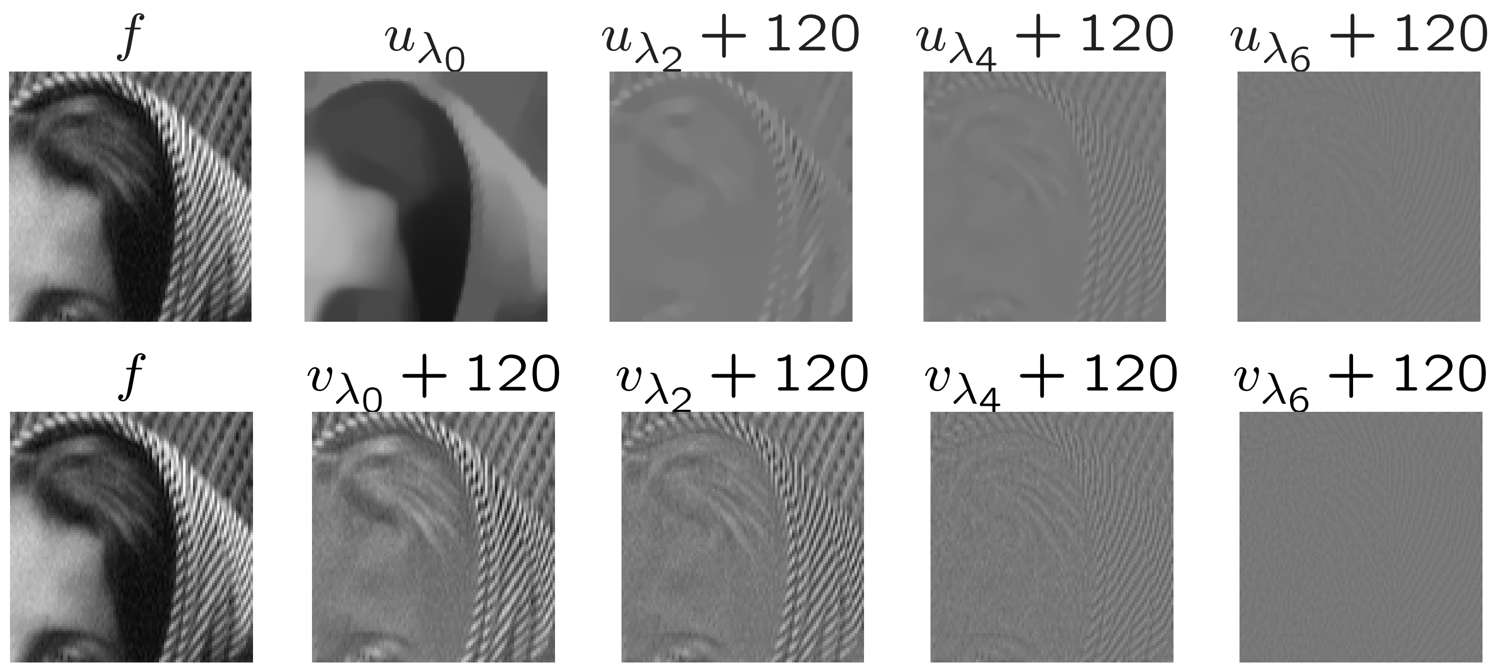
Successive decompositions of an image of a woman with $\lambda_0 = .0005$.

Barbara II



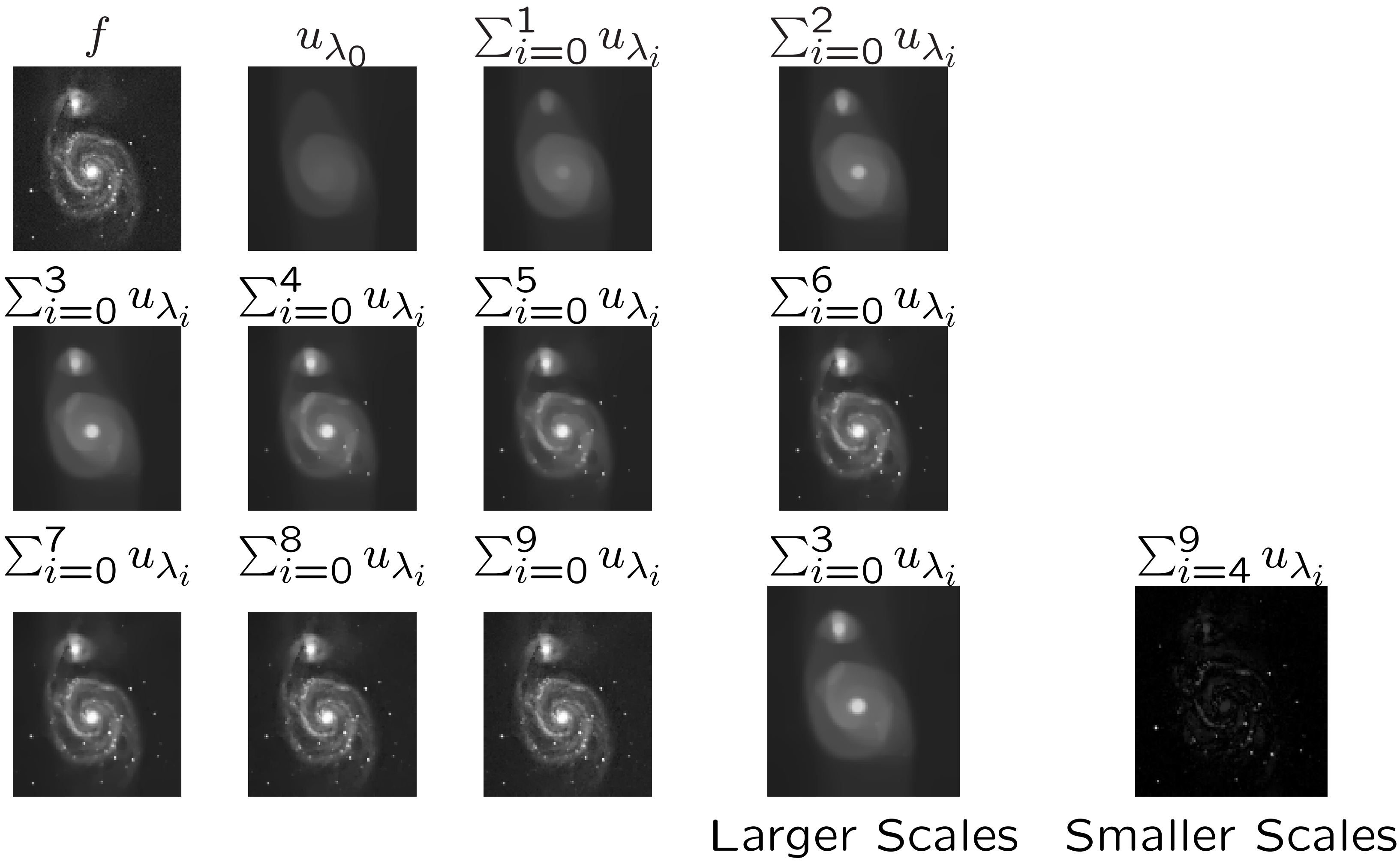
Decomposition of an initial image of a woman for 10 steps. Parameters: $\lambda_0 = .005$, and $\lambda_j = \lambda_0 2^j$.

Barbara: edges and texture



Representation of each u_j, v_j, for $0 \leq j \leq 6$. Parameters: $\lambda_0 = .005$, and $\lambda_j = \lambda_0 2^j$

Galaxy



Decomposition of an image of a galaxy for 10 steps. Parameters: $\lambda_0 = .001$, and $\lambda_j = \lambda_0 2^j$. The last two figures illustrate separation of scales.

Examples of (BV, L^2) expansions

$$f(x) = \alpha \chi_{B_R}(x) := \begin{cases} 1 & |x| \leq R \\ 0 & x \in \Omega \setminus B_R \end{cases}$$

$$u_\lambda = \left(\alpha - \frac{1}{\lambda R}\right)_+ \chi_{B_R} + \frac{1}{\lambda R}\frac{|B_R|}{|\Omega \setminus B_R|}\chi_{\Omega \setminus B_R}, \;\; v_\lambda := f - u_\lambda$$

Natural boundary condition, $\partial u_\lambda / \partial n_{|\partial\Omega} = 0$, implies $\int_\Omega v_\lambda dx = 0$

- No localization: non-zero constant of v_λ outside the ball B_R

The general hierarchical step then reads

$$\begin{aligned} u_j &= \left(\frac{1}{\lambda_{j-1}R} - \frac{1}{\lambda_j R}\right)\chi_{B_R} + \left(\frac{1}{\lambda_j R} - \frac{1}{\lambda_{j-1}R}\right)\frac{|B_R|}{|\Omega \setminus B_R|}\chi_{\Omega \setminus B_R} \\ v_j &= \frac{1}{\lambda_j R}\chi_{B_R} - \frac{1}{\lambda_j R}\frac{|B_R|}{|\Omega \setminus B_R|}\chi_{\Omega \setminus B_R} \sim 2^{-j} \text{ in } L^2 \ldots \end{aligned}$$

Localization

- The (BV, L^2) hierarchical expansion

$$\alpha\chi_{B_R}(x) \sim \sum_{j=0}^{k} u_j = \Big(\alpha - \frac{1}{\lambda_k R}\Big)\chi_{B_R} + \frac{1}{\lambda_k R}\frac{|B_R|}{|\Omega \setminus B_R|}\chi_{\Omega\setminus B_R}$$

$f - \sum^k u_j$ decays outside $supp(f)$

Strong convergence: geometrically vanishing error, $\|v_k\|_2 \sim \frac{1}{\lambda_k}$

Q. Localization: an image $f = g \bigoplus h$ with $supp(g) \cap supp(h) = \emptyset$;

assume $g \sim \sum g_j$ and $h \sim \sum h_j$.

What about $\sum g_j + h_j$ as a hierarchical expansion of f?

A. $\|f - \sum^k (g_j + h_j)\|_{W^{-1,\infty}} \leq \frac{1}{\lambda_k}$; quantify strong convergence;

The behavior of $supp(g_j)$ and $supp(h_j)$ relative to $supp(f)$.

- Localization: the spacial case $(g, h) = (f, 0)$

An example: $f = \chi_A(x) + p(2^N x)\chi_B(x), A \cap B = \emptyset$

$h \equiv h_N = p(2^N x)\chi_B(x)$ – the 'noisy part' of f with increasing N

$g = \chi_A(x)$ – the 'essential feature' in f.

If $2^N \gg \lambda$, then the 'u-component' of the $J(h, \lambda)$ minimizer fails to separate the essential part of h, since $\|h\|_{W^{-1,\infty}} \sim 2^{-N} < \frac{1}{2\lambda}$.

Need at least $k \sim N$ terms for $h \sim \sum^{k} h_j$ to remove the noisy part.

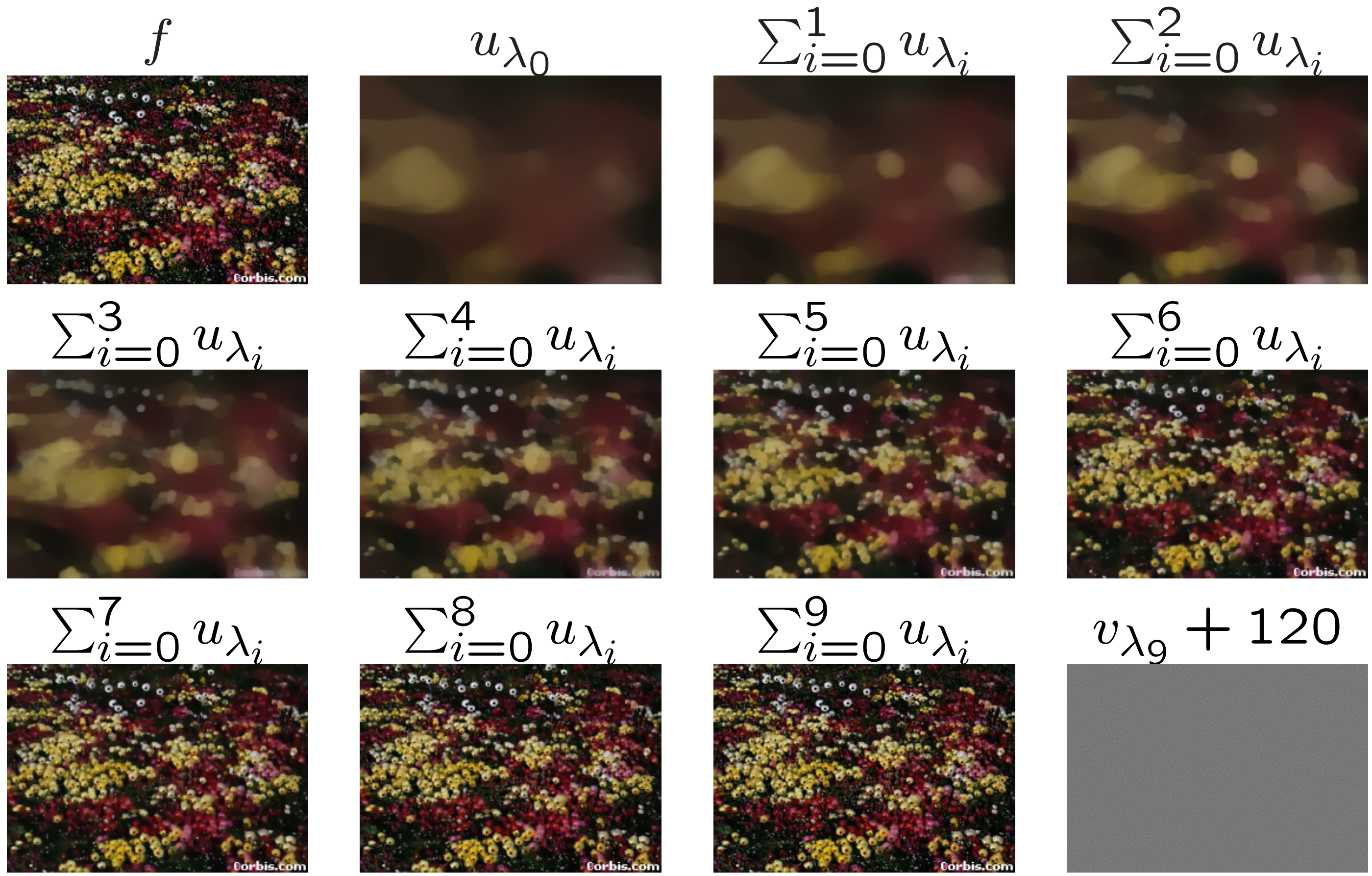
The expansion of g is independent of N: $\|g - \sum^{k} g_j\| \sim \frac{1}{\lambda_k}$

We are led to...

Q. How does $\sum^{k}(g_j + h_j)$ compare with $f \sim \sum^{k} u_j$?

A. To introduce a *localized* hierarchical expansion, adapted to the behavior of f in each subdomain.

Colored images



Decomposition of a vector-valued image of flowers for 10 steps.
Parameters: $\lambda_0 = .00025$, and $\lambda_j = \lambda_0 2^j$.

The '$u+v$' models and beyond

- The regularized Mumford-Shah by Ambrosio and Tortorelli:

$$\inf_{\{w,u,v \ | \ u+v=f\}} \left\{ \int_\Omega w^2 \Big[|\nabla u|^2 + |v|^2 \Big] dx + \lambda \Big[\varepsilon \|\nabla w\|^2_{L^2} + \frac{\|1-w\|^2_{L^2}}{\varepsilon} \Big] \right\}.$$

- Intermediate spaces $(L^2, B_1^{1,1})_\theta$:

extract and separate scales in terms of a wavelet decomposition $f = \sum \hat{f}_{jk}\psi_{jk}$.

- decomposition into *hierarchy* of dyadic scales

- Wavelet shrinkage based on a 'greedy' approach:

$$\text{DeVore – Lucier :} \qquad f \approx \sum_{|\hat{f}_{jk}| \geq \eta} \hat{f}_{jk}\psi_{jk}$$

- No such simple hierarchical description of $(L^2, BV)_\theta$

- But the smaller $B_1^{1,1}$ fails to capture sharp edges.

Ongoing work: beyond 'u+v' models

- Blurred images: K is a blurring kernel

$$J_K(f,\lambda;BV,L^2) := \inf_{u\in BV}\Big\{\lambda\|f-Ku\|^2_{L^2(\Omega)} + \|u\|_{BV(\Omega)}\Big\}.$$

$$f = Ku_0 + Ku_1 + ... + Ku_{k-1} + Ku_k + v_k, \qquad \|f\|_2^2 = \sum_{j=0}^{\infty}\Big[\frac{1}{\lambda_j}\|u_j\|_{BV} + \|Ku_j\|_2^2\Big]$$

- Multiplicative noise: $f = u_0 u_1 ... u_k \times v_k$

$$M(f,\lambda;BV,L^2) := \inf_{u\in BV_+(\Omega)}\Big\{\lambda\Big\|\frac{f}{u} - 1\Big\|^2_{L^2(\Omega)} + \|u\|_{BV(\Omega)}\Big\}.$$

- Hierarchical (SBV, L^2) decomposition (Ambrosio and Tortorelli)

$$AT^\varepsilon(f,\lambda) := \inf_{\{w,u,v \ | \ u+v=f\}}\Big\{\int_\Omega\Big[w^2|\nabla u|^2 + |v|^2\Big]dx + \lambda\Big[\varepsilon\|\nabla w\|^2_{L^2} + \frac{\|w-1\|^2_{L^2}}{\varepsilon}\Big]\Big\}.$$

Edge detectors $1-w_j = 1-w_{\lambda_j}$, supported along the boundaries enclosing the u_j's.

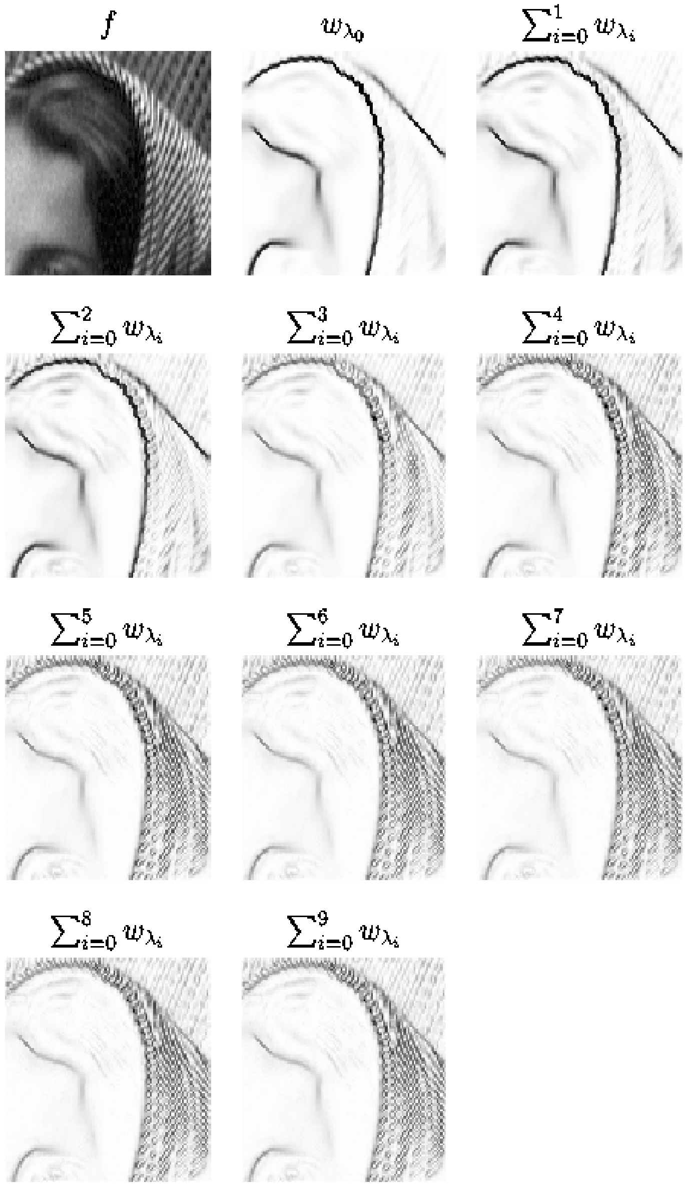
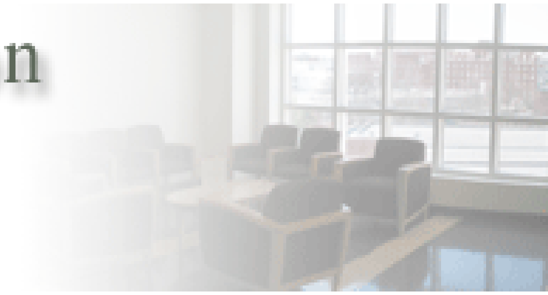


Figure 4.21: The sum of the w_i's using the Ambrosio-Tortorelli approximation of the image of a woman, using 10 steps. Parameters: $\beta_0 = .25$, $\alpha = 5$, $\rho = .0002$, and $\beta_k = 2^k \beta_0$



THANK YOU

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